

ITERATION OF CLOSED GEODESICS IN STATIONARY LORENTZIAN MANIFOLDS

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ABSTRACT. Following the lines of [8], we study the Morse index of the iterated of a closed geodesic in stationary Lorentzian manifolds, or, more generally, of a closed Lorentzian geodesic that admits a timelike periodic Jacobi field. Given one such closed geodesic γ , we prove the existence of a locally constant integer valued map Λ_γ on the unit circle with the property that the Morse index of the iterated γ^N is equal, up to a correction term $\epsilon_\gamma \in \{0, 1\}$, to the sum of the values of Λ_γ at the N -th roots of unity. The discontinuities of Λ_γ occur at a finite number of points of the unit circle, that are special eigenvalues of the linearized Poincaré map of γ . We discuss some applications of the theory.

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1. INTRODUCTION

It is well known that, unlike the Riemannian case, the geodesic action functional of a manifold endowed with a non positive definite metric tensor is always *strongly indefinite*, i.e., all its critical points have infinite Morse index. However, given a Lorentzian manifold

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(M, g) that has a timelike Killing vector field \mathcal{K} , one can consider a constrained geodesic variational problem whose critical points have finite Morse index ([7, 12, 13, 20]); the value of this index is computed in terms of a symplectic invariant related to the Conley–Zehnder index and the Maslov index of the linearized geodesic flow.

Following the classical Riemannian results, one wants to prove multiplicity results for closed geodesics using variational methods, including equivariant Morse theory. The closed geodesic variational problem is invariant by the action of the compact Lie group $O(2)$; every $O(2)$ -orbit of a closed geodesic contributes to the homology of the free loop space. In order to obtain multiplicity results, one needs to distinguish between critical orbits generated by the *tower of iterates* $(\gamma^N)_{N \geq 1}$ of the same geodesic γ . As proved by Gromoll and Meyer in the celebrated paper [14], fine estimates on the homological contribution of iterated closed geodesics can be given in terms of the Morse index and the nullity of the iterate. Thus, an essential step in the development of the Morse theory for closed geodesics is to establish the growth of the Morse index by iteration of a given closed geodesic. The deepest results in this direction for the Riemannian case are due to Bott in the famous paper [8]; using complexifications and a suitable intersection theory, Bott proves that all the information on the Morse index and the nullity of the iterates of a given closed Riemannian geodesic is encoded into two integer valued functions on the unit circle. Following Bott’s ideas, in this paper we will prove some iteration formulas for the Morse index of the critical points of the constrained variational problem for stationary closed Lorentzian geodesics mentioned above. More precisely, given a closed geodesic γ , we will show (Theorem 5.1) the existence of an integer valued function Λ_γ on the circle \mathbb{S}^1 with the property that, given a closed geodesic γ , its N -iterate γ^N has Morse index $\mu(\gamma^N)$ given by the sum of the values of Λ_γ at the N -th roots of unity, $k = 1, \dots, N$, plus a correction term $\epsilon_\gamma \in \{0, 1\}$. The difference $\mu_0(\gamma) = \mu(\gamma) - \epsilon_\gamma$, the *restricted Morse index* of γ , plays a central role in the theory; it can be interpreted as the index of the second variation of the geodesic action functional restricted to the space of variational vector fields arising from variations of γ by curves μ satisfying $g(\dot{\mu}, \mathcal{K}) = g(\dot{\gamma}, \mathcal{K})$ (constant).

In analogy with the Riemannian case, Λ_γ is a lower semi-continuous function on the circle (except possibly at 1 in a singular case mentioned below) and its jumps can occur only at points of \mathbb{S}^1 that belong to the spectrum of the (complexified) *linearized Poincaré* map \mathfrak{P}_γ of γ . Given one such discontinuity point $\rho \in \mathbb{S}^1$, the (complex) dimension of the kernel of $\mathfrak{P}_\gamma - \rho$ is an upper bound for the value of the jump of Λ_γ at ρ . Explicit, although extremely involved, computations can be attempted in order to compute the precise value of each jump of Λ_γ (see Subsection 4.2). It may be interesting to observe here that the question is reduced to an algebraic counting of the zeros in the spectrum (i.e., the *spectral flow*) of an *analytic* path of Fredholm self-adjoint operators, for which a finite dimensional reduction and a higher order method are available (see [11]).

As a special case of our iteration formula, we show that if γ is *strongly hyperbolic* (i.e., $\epsilon_\gamma = 0$ and there is no eigenvalue of \mathfrak{P}_γ on the unit circle), then the restricted Morse index of γ^N is equal to the restricted Morse index of γ multiplied by N . Also, the correction term ϵ_{γ^N} coincides with ϵ_γ for all N . As an application of this fact, we will use an argument from equivariant Morse theory to prove (Proposition 5.6) the existence of infinitely many *geometrically distinct* closed geodesics in a class of non simply connected globally hyperbolic stationary spacetimes, generalizing the results of [3].

A second important application of the theory developed in this paper is the proof of a *uniform linear growth* for the index of an iterate (Proposition 5.3); this is a crucial step in Gromoll and Meyer’s result on the existence of infinitely many closed geodesics in compact Riemannian manifolds. The uniform estimate on the linear growth allows to prove that the contribution to the homology of the free loop space in a fixed dimension provided by the tower of iterates of a given closed geodesic only depends on a uniformly bounded number of iterates.

Compared to the Riemannian case, several new phenomena appear in the stationary Lorentzian case. In first place, the question of the correction term ϵ_γ is somewhat puzzling, as this part of the index is not detected by the values of the function Λ_γ on $\mathbb{S}^1 \setminus \{1\}$. Its geometric interpretation is a little involved; roughly speaking, ϵ_γ vanishes when γ can be perturbed to a curve with less energy only by curves that “form a fixed angle” with the timelike Killing field \mathcal{K} . Infinitesimally, this amounts to saying that the index of the index form does not decrease when the form is restricted to the space of variations V satisfying $\mathbf{g}(V', \mathcal{K}) - \mathbf{g}(V, \mathcal{K}') = 0$, where the prime denotes covariant differentiation along γ . Particularly significative is the fact that the correction term ϵ_γ is the same for all the geodesics in the tower of iterates of γ . An important class of examples of geodesics γ with $\epsilon_\gamma = 0$ is obtained by taking geodesics that are everywhere orthogonal to \mathcal{K} in the *static* case, i.e., when the orthogonal distribution \mathcal{K}^\perp to \mathcal{K} is integrable (see Example 5.1). In this case, every integral submanifold of \mathcal{K}^\perp is a Riemannian totally geodesic hypersurface of M ; this suggests that ϵ_γ can be interpreted as a sort of measure of the “non Riemannian behavior” of the closed geodesic γ . It is plausible to conjecture that the question of the distribution of conjugate points along γ be related to the value of ϵ_γ ; the results of an investigation in this direction are left to a forthcoming paper.

A second peculiar phenomenon of the stationary Lorentzian case is the existence of a singular class of closed geodesics γ , that are characterized by the fact that the covariant derivative \mathcal{K}' along γ is pointwise multiple of the projection of \mathcal{K} onto the orthogonal space $\dot{\gamma}^\perp$; this includes in particular all closed geodesics along which the Killing field is parallel. As it is shown in [16], the fact that \mathcal{K} is singular along γ is equivalent to the existence of a family of parallel vectorfields along the geodesics that generate $\mathcal{K}(s)^\perp$ for every $s \in [0, 1]$. This is true in particular when the geodesic is contained in a totally geodesic hypersurface orthogonal to \mathcal{K} . Again, orthogonal geodesics are singular in the static case. When γ is singular, the question of semi-continuity of the function Λ_γ is more delicate, and, in fact, it may fail to hold at the point 1 if $\epsilon_\gamma = 1$, even when the linearized Poincaré map of γ does not contain 1 in its spectrum.

As already observed in [8], in spite of the initial geometrical motivation the theory developed in the present paper is better cast in the language of Morse–Sturm differential systems in the complex space \mathbb{C}^n . We will first discuss our results in this abstract setup (Section 2); the reduction of the stationary Lorentzian geodesic problem to a Morse–Sturm system is done via a parallel (not necessarily periodic) trivialization of the orthogonal bundle $\dot{\gamma}^\perp$ along the geodesic γ (Subsection 5.2). As to this point, it is interesting to observe here that, if on one hand to use a parallel trivialization simplifies the corresponding Morse–Sturm system and its index form (see (2.1), (2.7)), on the other hand the lack of periodicity of such trivializations imposes the introduction of more involved boundary conditions. In our notations, information on the boundary conditions is encoded in the endomorphism T (see Subsection 2.1), which represents the parallel transport along the geodesic. When the geodesic is orientable, then T is (the complex extension of an isomorphism) in the connected component of the identity of $\mathrm{GL}(\mathbb{R}^{2n})$. Using this setup, no distinction is necessary between orientable and non orientable closed geodesics; recall that these two cases are distinguished in Bott’s original work, the non orientable ones corresponding to the N -th roots of -1 . We observe that the theory developed in this work can be set in a more general background than stationary spacetimes, that is, when considering geodesics in general Lorentzian manifolds admitting a periodic timelike Jacobi field.

The paper is organized as follows. The main technical results, presented in the context of abstract Morse–Sturm systems in \mathbb{C}^n , are discussed in the first part of the paper (Sections 2, 3 and 4), while the applications to closed geodesics are discussed in Section 5. In Section 2 we set up the basis of the theory, with a description of a class of complex Morse–Sturm systems that are symmetric with respect to a nondegenerate Hermitian form of index 1 in \mathbb{C}^n , their index form, and with the description of two families of closed

subspaces of the Sobolev space $H^1([0, 1], \mathbb{C}^n)$. These spaces are parameterized by unit complex numbers, and a central point is determining their continuity with respect to the parameter. Although only the continuity of these spaces is actually required in our theory, we will show that the dependence is in fact analytic. The purpose of this fact is that, in view to future developments, one might attempt to use higher order methods for determining the value of the jumps of the index function (see Subsection 4.2), which require analyticity of the eigenvalues and of the eigenvectors of the corresponding self-adjoint operators (see [11]). At this stage, this seems to be a rather involved question, that will be treated only marginally in this paper.

In Section 3 we use a functional analytical approach to determine the kernel of the index form restricted to the family of closed subspaces mentioned above. Section 4 contains the main technical result of the paper (Proposition 4.2), which is a formula relating the index of the N -iterate with the sum of the indexes of the index form at the space of vector fields satisfying boundary conditions involving the N -th roots of unity. Following Bott's suggestive terminology, we call this the *Fourier theorem*. In Section 5 the theory is applied to the case of closed geodesics in stationary Lorentzian manifolds, with some emphasis to the static case which provides interesting examples of singular solutions. Propositions 5.3 and 5.6 are two direct applications of our results to the global theory of closed Lorentzian geodesics, which is the final objective of our research. Finally, we conclude with a short section containing remarks, conjectures and suggestions for future developments.

2. ON A CLASS OF NON POSITIVE DEFINITE MORSE–STURM SYSTEMS IN \mathbb{C}^n

2.1. The basic setup. Let us consider given the following objects:

- (a) n is an integer greater than or equal to 1
- (b) g is a nondegenerate symmetric bilinear form on \mathbb{R}^n having index 1, extended by sesquilinearity to a nondegenerate Hermitian form on \mathbb{C}^n ;
- (c) $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a g -preserving linear isomorphism of \mathbb{R}^n , extended by complex linearity to a g -preserving isomorphism of \mathbb{C}^n ;
- (d) $[0, 1] \ni t \mapsto R(t) \in \text{End}(\mathbb{R}^n)$ is a continuous map of g -symmetric (i.e., $gR(t) = R(t)^*g$) linear endomorphisms of \mathbb{R}^n satisfying:

$$R(0)T = TR(1);$$

$R(t)$ is extended by complex linearity to a g -Hermitian endomorphism of \mathbb{C}^n .

- (e) $Y : [0, 1] \rightarrow \mathbb{R}^n \subset \mathbb{C}^n$ is a C^2 -solution of the *Morse–Sturm system*:

$$(2.1) \quad V''(t) = R(t)V(t)$$

that satisfies:

- (e1) $g(Y, Y) < 0$ everywhere on $[0, 1]$;
- (e2) $TY(1) = Y(0)$ and $TY'(1) = Y'(0)$.

The solution Y of (2.1) will be called *singular* if $Y'(s)$ is a multiple of $Y(s)$ for all $s \in [0, 1]$; the singularity of Y is equivalent to either one of the following two conditions:

$$(2.2) \quad g(Y, Y)Y' = g(Y', Y)Y, \quad \text{or} \quad \left[\frac{Y}{g(Y, Y)} \right]' + \frac{Y'}{g(Y, Y)} = 0 \quad \text{on } [0, 1].$$

The g -symmetry of R implies that, given any two solutions V_1 and V_2 of (2.1), then the quantity $g(V_1', V_2) - g(V_1, V_2')$ is constant on $[0, 1]$:

$$(2.3) \quad \frac{d}{ds} [g(V_1', V_2) - g(V_1, V_2')] = g(V_1'', V_2) - g(V_1, V_2'') = g(RV_1, V_2) - g(V_1, RV_2) = 0.$$

We will consider extensions to the real line $Y : \mathbb{R} \rightarrow \mathbb{C}^n$ and $R : \mathbb{R} \rightarrow \text{End}(\mathbb{C}^n)$ of the maps Y and R above by setting:

$$(2.4) \quad Y(t + N) = T^{-N}Y(t), \quad R(t + N) = T^{-N}R(t)T^N, \quad \forall t \in [0, 1[, \quad N \in \mathbb{Z};$$

having this in mind, we also set $R_N(t) = R(tN)$ and $Y_N(t) = Y(tN)$ for all $t \in [0, 1]$. Observe that Y_N is of class C^2 and R_N is continuous on $[0, 1]$, moreover, from (2.4) one gets easily:

$$(2.5) \quad Y_N(t + k/N) = T^{-k}Y_N(t), \quad R_N(t + k/N) = T^{-k}R_N(t)T^k,$$

for every $k \in \mathbb{Z}$. The N -th iterated of the Morse–Sturm system (2.1) is the Morse–Sturm system:

$$(2.6) \quad V''(t) = N^2 R_N(t) V(t).$$

If Y is a singular solution of (2.1), then Y_N is a singular solution of (2.6), in which case equalities (2.2) hold with Y replaced by Y_N . We will consider the following additional data.

2.2. The index forms. Let \mathcal{H} be the Hilbert space $H^1([0, 1], \mathbb{C}^n)$ of \mathbb{C}^n -valued maps on the interval $[0, 1]$ and of Sobolev class H^1 ; moreover, for all $N \geq 1$ let $I_N : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ be the bounded sesquilinear form:

$$(2.7) \quad I_N(V, W) = \int_0^1 [g(V', W') + N^2 g(R_N V, W)] dt, \quad V, W \in \mathcal{H}.$$

We will also introduce a smooth family of positive definite Hermitian forms g_t^N on \mathbb{C}^n , defined using Y by:

$$(2.8) \quad g_t^N(V, W) = g(V, W) - 2 \frac{g(V, Y_N(t)) \cdot g(W, Y_N(t))}{g(Y_N(t), Y_N(t))};$$

denote by $A : [0, 1] \times \mathbb{N} \rightarrow \mathcal{L}(\mathbb{C}^n)$ the smooth family of symmetric isomorphisms such that

$$(2.9) \quad g(V, W) = g_t^N(A(t, N)V, W)$$

for every $V, W \in \mathbb{C}^n$. We will think of \mathcal{H} endowed with a family of Hilbert space inner products:

$$(2.10) \quad \langle V, W \rangle_N = \int_0^1 [g_t^N(V', W') + g_t^N(V, W)] dt.$$

2.3. Analytic families of closed subspaces. We will now define a family of closed subspaces of \mathcal{H} , as follows. Let \mathbb{S}^1 denote the set of unit complex numbers; for $\rho \in \mathbb{S}^1$ and $N \geq 1$, set:

$$\mathcal{H}^\rho(N) = \{V \in \mathcal{H} : T^N V(1) = \rho^N V(0)\},$$

$$\mathcal{H}_*(N) = \left\{ V \in \mathcal{H} : g(V', Y_N) - g(V, Y'_N) = C_V \text{ (constant) a.e. on } [0, 1] \right\},$$

and:

$$\mathcal{H}_0(N) = \left\{ V \in \mathcal{H} : g(V', Y_N) - g(V, Y'_N) = 0 \text{ a.e. on } [0, 1] \right\}.$$

Finally, define:

$$\mathcal{H}_*^\rho(N) = \mathcal{H}_*(N) \cap \mathcal{H}^\rho(N), \quad \mathcal{H}_0^\rho(N) = \mathcal{H}_0(N) \cap \mathcal{H}^\rho(N).$$

Proposition 2.1. *The kernel of the restriction of I_N to $\mathcal{H}^\rho(N)$ coincides with the kernel of the restriction of I_N to $\mathcal{H}_*^\rho(N)$, and it is given by the finite dimensional space:*

$$(2.11) \quad \left\{ V \in C^2([0, 1], \mathbb{C}^n) : V \text{ solution of (2.6), } T^N V(1) = \rho^N V(0), T^N V'(1) = \rho^N V'(0) \right\}.$$

Proof. That (2.11) is the kernel of I_N in $\mathcal{H}^\rho(N)$ follows easily from a partial integration in (2.7). Denote by $\mathfrak{V}(N)$ the subspace of $\mathcal{H}^\rho(N)$ consisting of vector fields of the form $f \cdot Y_N$, where $f \in H^1([0, 1], \mathbb{C}^n)$ is such that $f(0) = f(1) = 0$. The conclusion follows easily from the fact that $\mathcal{H}^\rho(N) = \mathcal{H}_*^\rho(N) + \mathfrak{V}^\rho(N)$, that the spaces $\mathcal{H}_*^\rho(N)$ and $\mathfrak{V}^\rho(N)$ are I_N -orthogonal, that I_N is negative definite on $\mathfrak{V}(N)$, and that (2.11) is contained in $\mathcal{H}_*^\rho(N)$. \square

Corollary 2.2. *If ρ is not an N -th root of unity, then $\text{Ker}(I_N|_{\mathcal{H}_*^\rho(N) \times \mathcal{H}_*^\rho(N)}) \subset \mathcal{H}_0^\rho(N)$.*

Proof. Recalling that T is g -symmetric, we have:

$$\begin{aligned} g(V'(0), Y_N(0)) - g(V(0), Y'_N(0)) &= g(V'(1), Y_N(1)) - g(V(1), Y'_N(1)) \\ &= \rho^N [g(V'(0), Y_N(0)) - g(V(0), Y'_N(0))], \end{aligned}$$

from which the conclusion follows. \square

Let us introduce the following:

Definition 2.3. Let \mathfrak{H} be a complex Hilbert space, $I \subset \mathbb{R}$ an interval and $\{\mathcal{D}^t\}_{t \in I}$ be a family of closed subspaces of \mathfrak{H} . We say that $\{\mathcal{D}^t\}_{t \in I}$ is a C^k -family, $k = 0, \dots, \infty$, (resp., an *analytic family*) of subspaces if for all $t_0 \in I$ there exist $\varepsilon > 0$, a C^k (resp., an analytic) curve $\alpha :]t_0 - \varepsilon, t_0 + \varepsilon[\cap I \mapsto \mathcal{L}(\mathfrak{H})$ and a closed subspace $\overline{\mathcal{D}} \subset \mathfrak{H}$ such that $\alpha(t)$ is an isomorphism and $\alpha(t)(\mathcal{D}^t) = \overline{\mathcal{D}}$ for all t .

Definition 2.3 is generalized obviously to the case of families $\{\mathcal{D}^\theta\}_{\theta \in \mathbb{S}^1}$ parameterized on the circle. Let us give a criterion for the smoothness of a family of closed subspaces:

Proposition 2.4. *Let $I \subset \mathbb{R}$ be an interval, $\mathfrak{H}, \tilde{\mathfrak{H}}$ be Hilbert spaces and $F : I \mapsto \mathcal{L}(\mathfrak{H}, \tilde{\mathfrak{H}})$ be a C^k (resp., analytic) map such that each $F(t)$ is surjective. Then, the family $\mathcal{D}^t = \text{Ker}(F(t))$ is a C^k -family (resp., an analytic family) of closed subspaces of \mathfrak{H} .*

Proof. See for instance [12, Lemma 2.9]. \square

Clearly, if $\{\mathcal{D}^t\}_{t \in I}$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H}_1 and \mathfrak{H}_1 is a closed subspace of \mathfrak{H} , then $\{\mathcal{D}^t\}_{t \in I}$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H} . It is also clear that, given a C^k (resp., an analytic) family of closed subspaces $\{\mathcal{D}^t\}_{t \in I}$ of \mathfrak{H} and given a C^k (resp., an analytic) map $t \mapsto \psi_t$ of isomorphisms of \mathfrak{H} , then $\{\psi_t(\mathcal{D}^t)\}_{t \in I}$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H} . We will need later a slight improvement of Proposition 2.4:

Corollary 2.5. *If $\{\mathcal{D}^t\}_{t \in I}$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H} , and if $F : I \mapsto \mathcal{L}(\mathfrak{H}, \tilde{\mathfrak{H}})$ is a C^k (resp., an analytic) map such that the restriction of $F(t)$ to \mathcal{D}^t is surjective for all $t \in I$, then $\mathcal{E}_t = \text{Ker}(F(t)) \cap \mathcal{D}^t$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H} .*

Proof. Let $\alpha :]t_0 - \varepsilon, t_0 + \varepsilon[\cap I \mapsto \mathcal{L}(\mathfrak{H})$ be a local trivialization of $\{\mathcal{D}^t\}$, $\alpha(t)(\mathcal{D}^t) = \overline{\mathcal{D}}$ and consider the C^k (analytic) map $t \mapsto \tilde{F}(t) : \overline{\mathcal{D}} \rightarrow \tilde{\mathfrak{H}}$ given by $\tilde{F}(t) = F(t) \circ (\alpha(t)|_{\mathcal{D}^t})^{-1}$. Since the restriction of $F(t)$ to \mathcal{D}^t is surjective, then $\tilde{F}(t)$ is surjective. By Proposition 2.4, the family $\text{Ker}(\tilde{F}(t)) = \alpha(t)(\mathcal{E}_t)$ is a C^k (resp., an analytic) family of closed subspaces of $\overline{\mathcal{D}}$. It follows that $\mathcal{E}_t = \alpha(t)^{-1}[\text{Ker}(\tilde{F}(t))]$ is a C^k (resp., an analytic) family of closed subspaces of \mathfrak{H} . \square

Proposition 2.6. *For all $N \geq 1$, the collection $\{\mathcal{H}_*^\rho(N)\}_{\rho \in \mathbb{S}^1}$ is an analytic family of closed subspaces of \mathcal{H} . If Y is not singular, then the same conclusion holds also for the family $\{\mathcal{H}_0^\rho(N)\}_{\rho \in \mathbb{S}^1}$.*

Proof. Consider the analytic map $\mathbb{S}^1 \ni \rho \mapsto F_\rho, F_\rho : \mathcal{H} \rightarrow \mathbb{C}^n$ given by $F_\rho(V) = T^N V(1) - \rho^N V(0)$; in order to apply Proposition 2.4 we need to show that the restriction of F_ρ to $\mathcal{H}_*^\rho(N)$ is surjective for all $\rho \in \mathbb{S}^1$. Clearly, $F_\rho : \mathcal{H} \rightarrow \mathbb{C}^n$ is surjective. Given $W \in \mathcal{H}$, there exists $V \in \mathcal{H}_*^\rho(N)$ with $V(0) = W(0)$ and $V(1) = W(1)$. Such V is obtained by setting $V(s) = W(s) + f_W(s) \cdot Y_N(s)$, $s \in [0, 1]$, where

$$f_W(s) = \int_0^s \frac{C + g(W, Y'_N) - g(W', Y_N)}{g(Y_N, Y_N)} dr$$

and

$$(2.12) \quad C = \left(\int_0^1 \frac{dr}{g(Y_N, Y_N)} \right)^{-1} \int_0^1 \frac{g(W', Y_N) - g(W, Y'_N)}{g(Y_N, Y_N)} dr.$$

It is easily seen that such V satisfies $g(V', Y_N) - g(V, Y'_N) \equiv C$, the obvious details of such computation being omitted. Since in this situation $F_\rho(V) = F_\rho(W)$, it follows immediately that also the restriction of F_ρ to $\mathcal{H}_*^\rho(N)$ is surjective.

Consider now the case of the family $\mathcal{H}_0^\rho(N)$; the non singularity assumption on Y implies that we can find a subinterval $[a, b] \subset]0, 1[$ such that

$$(2.13) \quad \mathbf{T}_N := \left[\frac{Y_N}{g(Y_N, Y_N)} \right]' + \frac{Y'_N}{g(Y_N, Y_N)} \neq 0 \quad \text{on } [a, b].$$

In this case, we show that the restriction of F_ρ to $\mathcal{H}_0^\rho(N)$ is surjective by showing that for all $Z_0, Z_1 \in \mathbb{C}^n$ there exists $W \in \mathcal{H}$ with $W(0) = Z_0$, $W(1) = Z_1$ and such that the quantity C given in (2.12) vanishes. For, choose arbitrary smooth maps $t_1 : [0, a] \rightarrow \mathbb{C}^n$ and $t_2 : [b, 1] \rightarrow \mathbb{C}^n$ such that $t_1(0) = Z_0$, $t_2(1) = Z_1$ and $t_1(a) = t_2(b) = 0$. The desired W is then obtained by setting:

$$W(s) = \begin{cases} t_1(s) & \text{if } s \in [0, a[, \\ h(s) & \text{if } s \in [a, b], \\ t_2(s) & \text{if } s \in]b, 1], \end{cases}$$

where $h \in H_0^1([a, b]; \mathbb{C}^n)$ is to be chosen in such a way that

$$(2.14) \quad \int_a^b \frac{g(h', Y_N) - g(h, Y'_N)}{g(Y_N, Y_N)} dr = - \int_0^a \frac{g(t'_1, Y_N) - g(t_1, Y'_N)}{g(Y_N, Y_N)} dr - \int_b^1 \frac{g(t'_2, Y_N) - g(t_2, Y'_N)}{g(Y_N, Y_N)} dr.$$

The left hand side of this equality defines a bounded linear functional on $H_0^1([a, b], \mathbb{C}^n)$ which is not null; this is easily seen using partial integration:

$$\int_a^b \frac{g(h', Y_N) - g(h, Y'_N)}{g(Y_N, Y_N)} dr = - \int_a^b g(h, \mathbf{T}_N) dr,$$

and using our assumption (2.13). In particular, a function $h \in H_0^1([a, b], \mathbb{C}^n)$ satisfying (2.14) can be found, which concludes the argument. \square

2.4. Singular solutions. Let us now assume that Y is a singular solution of the Morse–Sturm system (2.1), which is equivalent to assuming that the maps \mathbf{T}_N defined in (2.13) vanish identically on $[0, 1]$ for all $N \geq 1$.

Lemma 2.7. *If Y is a singular solution of (2.1) and $\rho \in \mathbb{S}^1$ is an N -th root of unity, then $\mathcal{H}_0^\rho(N) = \mathcal{H}_*^\rho(N)$.*

Proof. If $V \in \mathcal{H}_*^\rho(N)$, a direct computation gives:

$$(2.15) \quad C_V \int_0^1 \frac{dr}{g(Y_N, Y_N)} = \int_0^1 \frac{g(V', Y_N) - g(V, Y'_N)}{g(Y_N, Y_N)} dr$$

$$= \frac{g(V, Y_N)}{g(Y_N, Y_N)} \Big|_0^1 - \int_0^1 g(V, \mathbf{T}_N) dr = \frac{g(V, Y_N)}{g(Y_N, Y_N)} \Big|_0^1 = (\rho^N - 1) \frac{g(V(0), Y_N(0))}{g(Y_N(0), Y_N(0))},$$

from which the conclusion follows easily. \square

Let us show now that the N -th roots of unity are the unique discontinuities of the family $\{\mathcal{H}_0^\rho(N)\}$.

Proposition 2.8. *If Y is a singular solution of (2.1) and $\mathcal{A} \subset \mathbb{S}^1$ is a connected subset that does not contain any N -th root of unity, the family $\{\mathcal{H}_0^\rho(N)\}_{\rho \in \mathcal{A}}$ is an analytic family of closed subspaces of \mathcal{H} .*

Proof. We use Corollary 2.5 applied to the analytic family $\mathcal{H}_*^\rho(N)$ and the constant map $F(t) = F_N : \mathcal{H} \rightarrow \mathbb{C}$ defined by:

$$F_N(V) = \left(\int_0^1 \frac{dr}{g(Y_N, Y_N)} \right)^{-1} \int_0^1 \frac{g(V', Y_N) - g(V, Y'_N)}{g(Y_N, Y_N)} dr.$$

The restriction of F_N to $\mathcal{H}_*^\rho(N)$ is the map $V \mapsto C_V$; by (2.15), such restriction is surjective (i.e., not identically zero) when ρ is not an N -th root of unity. Observe indeed that, as it follows easily arguing as in the proof of Proposition 2.6, $V(0)$ is an arbitrary vector of \mathbb{C}^n when V varies in $\mathcal{H}_*^\rho(N)$. \square

In particular, we have the following:

Corollary 2.9. *If Y is a singular solution of (2.1), then the collection $\{\mathcal{H}_0^\rho(1)\}_{\rho \in \mathbb{S}^1 \setminus \{1\}}$ is an analytic family of closed subspaces of \mathcal{H} .* \square

2.5. Finiteness of the index.

Proposition 2.10. *For all $N \geq 1$ and for all $\rho \in \mathbb{S}^1$, the restriction of I_N to $\mathcal{H}_*^\rho(N) \times \mathcal{H}_*^\rho(N)$ is essentially positive, i.e., it is represented (relatively to the inner product (2.10)) by a self-adjoint operator on $\mathcal{H}_*^\rho(N)$ which is a compact perturbation of a positive isomorphism. In particular, the index of I_N on $\mathcal{H}_*^\rho(N)$ is finite.*

Proof. We will show that the restriction of I_N to $\mathcal{H}_*^\rho(N)$ is the sum of the inner product $\langle \cdot, \cdot \rangle_N$ and a symmetric bilinear form B which is continuous relatively to the C^0 -topology. The conclusion will follow from the fact that the inclusion of H^1 into C^0 is compact, and therefore B is represented by a compact operator.

The linear map $\mathcal{H}_*^\rho(N) \ni V \mapsto C_V \in \mathbb{C}$ is continuous relatively to the C^0 -topology, for:

$$C_V = (\rho^N - 1)g(V(0), Y_N(0)) - 2 \int_0^1 g(V, Y'_N) dt.$$

A straightforward calculation shows that, for $V, W \in \mathcal{H}_*^\rho(N)$, $I_N(V, W)$ can be written as:

$$I_N(V, W) = \langle V, W \rangle_N + \int_0^1 \left[2 \frac{[g(V, Y'_N) + C_V] \cdot [g(W, Y'_N) + C_W]}{g(Y_N, Y_N)} - g_t^N(V, W) \right] dt,$$

from which the conclusion follows easily. \square

Corollary 2.11. *Let $\mathcal{A} \subset \mathbb{S}^1$ be a connected subset such that the restriction of I_N to $\mathcal{H}_*^\rho(N)$ is nondegenerate for all $\rho \in \mathcal{A}$. Then, the index of such restriction is constant on \mathcal{A} . Similarly, if Y is not a singular solution of (2.1), the same result holds for the restriction of I_N to $\mathcal{H}_0^\rho(N)$; if Y is a singular solution of (2.1), then the result holds under the additional assumption that \mathcal{A} does not contain any N -th root of unity.*

Proof. By continuity, the jumps of the map $\mathbb{S}^1 \ni \rho \mapsto \mathfrak{n}_-(I_N|_{\mathcal{H}_*^\rho(N) \times \mathcal{H}_*^\rho(N)}) \in \mathbb{N}$ can only occur at those points ρ where I_N is degenerate on $\mathcal{H}_*^\rho(N)$. The case of $\mathcal{H}_0^\rho(N)$ is analogous, using Proposition 2.8. \square

The discontinuities of the index function will be studied in Subsection 4.2 below.

2.6. The linear Poincaré map. The last ingredient of our theory is the linear map $\mathfrak{P} : \mathbb{C}^n \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n \oplus \mathbb{C}^n$ defined by:

$$\mathfrak{P}(v, w) = (TJ(1), TJ'(1)),$$

where $J : [0, 1] \rightarrow \mathbb{C}^n$ is the (unique) solution of the Morse–Sturm system (2.1) satisfying the initial conditions $J(0) = v$ and $J'(0) = w$. Using (2.4) one sees immediately that, given $N \geq 1$, the N -th power \mathfrak{P}^N is given by:

$$\mathfrak{P}^N(v, w) = (T^N J(1), T^N J'(1)),$$

where J is the solution of the equation $J'' = N^2 R_N J$ satisfying $J(0) = v$ and $J'(0) = w$. We will call \mathfrak{P} the *linear Poincaré map* of the Morse–Sturm system (2.1); clearly, \mathfrak{P} is the complex linear extension of an endomorphism of $\mathbb{R}^n \oplus \mathbb{R}^n$ defined using the real Morse–Sturm system. In particular, the spectrum $\mathfrak{s}(\mathfrak{P})$ of \mathfrak{P} is closed by conjugation.

Proposition 2.12. *For all $\rho \in \mathbb{S}^1$ and all $N \geq 1$, the map $V \mapsto (V(0), V'(0))$ gives an isomorphism from the kernel of the restriction of I_N to $\mathcal{H}_*^\rho(N)$ onto the ρ^N -eigenspace of \mathfrak{P}^N .*

Proof. It follows immediately from Proposition 2.1. \square

The *restricted* linear Poincaré map \mathfrak{P}_0 is the complex linear extension of the restriction of \mathfrak{P} to the invariant subspace $\mathfrak{J}_0 \subset \mathbb{R}^n \oplus \mathbb{R}^n$ defined by:

$$\mathfrak{J}_0 = \{(v, w) \in \mathbb{R}^n \oplus \mathbb{R}^n : g(w, Y(0)) - g(v, Y'(0)) = 0\}.$$

The invariance of \mathfrak{J}_0 is easily established using (2.3) and the equalities $Y(0) = TY(1)$, $Y'(0) = TY'(1)$. Clearly, $\mathfrak{s}(\mathfrak{P}_0) \subset \mathfrak{s}(\mathfrak{P})$; actually, the following holds:

Lemma 2.13. $\mathfrak{s}(\mathfrak{P}_0) \setminus \{1\} = \mathfrak{s}(\mathfrak{P}) \setminus \{1\}$.

Proof. As in Corollary 2.2. \square

2.7. The index sequences and the nullity sequences. Recall that, given a symmetric bilinear form $B : V \times V \rightarrow \mathbb{R}$ on a real vector space, the index and the nullity of B are defined respectively as the dimension of a maximal subspace on which B is negative definite, and the dimension of the kernel of B . Similarly, one defines index and nullity of a Hermitian sesquilinear bilinear form on a complex vector space; the index and the nullity of a symmetric bilinear form on a real vector space V are equal respectively to the index and the nullity of the sesquilinear extension of B to the complexification of V .

Definition 2.14. For all $\rho \in \mathbb{S}^1$, define the sequences:

$$\begin{aligned} \lambda_*(\rho, N) &= \text{index of } I_N \text{ on } \mathcal{H}_*^\rho(N), & \lambda_0(\rho, N) &= \text{index of } I_N \text{ on } \mathcal{H}_0^\rho(N), \\ \nu_*(\rho, N) &= \text{nullity of } I_N \text{ on } \mathcal{H}_*^\rho(N), & \nu_0(\rho, N) &= \text{nullity of } I_N \text{ on } \mathcal{H}_0^\rho(N), \end{aligned}$$

where $N \geq 1$.

Clearly:

$$\lambda_0(\rho, N) \leq \lambda_*(\rho, N) \leq \lambda_0(\rho, N) + 1$$

for all $N \geq 1$ and all $\rho \in \mathbb{S}^1$. By Corollary 2.2, we have $\nu_*(\rho, N) \leq \nu_0(\rho, N)$ when ρ is not an N -th root of unity; we will show later (Corollary 3.5) that $\nu_0(\rho, 1) \leq \nu_*(\rho, 1)$. Corollary 2.11 above says that the maps $\rho \mapsto \lambda_*(\rho, N)$ are constant on connected subsets of the circle where $\nu_*(\rho, N)$ vanishes.

The theory developed so far gives us the following properties of the index and the nullity sequences:

Proposition 2.15. *For all $N \geq 1$, the following statements hold.*

- (a) *The map $\rho \mapsto \lambda_*(\rho, N)$ is lower semi-continuous on \mathbb{S}^1 , and so is $\rho \mapsto \lambda_0(\rho, N)$ if Y is not a singular solution of (2.1); if Y is a singular solution, then the map $\rho \mapsto \lambda_0(\rho, N)$ is lower semi-continuous on every connected component of \mathbb{S}^1 that does not contain N -th roots of unity.*
- (b) $\nu_*(\rho, N) = \dim(\text{Ker}(\mathfrak{P}^N - \rho^N))$.
- (c) $\lambda_*(\rho, N) = \lambda_*(\bar{\rho}, N)$, $\nu_*(\rho, N) = \nu_*(\bar{\rho}, N)$, $\lambda_0(\rho, N) = \lambda_*(\bar{\rho}, N)$, and $\nu_0(\rho, N) = \nu_*(\bar{\rho}, N)$.

Proof. The lower continuity of λ_* and λ_0 claimed in part (a) follows easily from the continuity of the family of subspaces $\mathcal{H}_*^\rho(N)$ and $\mathcal{H}_0^\rho(N)$, which was proved in Proposition 2.6 for the non singular case, and in Proposition 2.8 for the singular case. Part (b) is a re-statement of Proposition 2.12. For part (c), it suffices to observe that the map $V \mapsto \bar{V}$ (pointwise complex conjugation) sends isomorphically the kernel (resp., a maximal negative subspace) of $I_N|_{\mathcal{H}_*^\rho(N) \times \mathcal{H}_*^\rho(N)}$ onto the kernel (resp., a maximal negative subspace) of $I_N|_{\mathcal{H}_*^{\bar{\rho}}(N) \times \mathcal{H}_*^{\bar{\rho}}(N)}$. Similarly for $I_N|_{\mathcal{H}_0^\rho(N) \times \mathcal{H}_0^\rho(N)}$. \square

3. ON THE NULLITY SEQUENCES

The aim of this section is to study the kernel of the restriction of the bilinear form $I = I_1$ to the space \mathcal{H}_0^ρ . We will perform this task by determining a differential equation satisfied by vector fields V_ρ that are eigenvectors of the restriction of I_1 to \mathcal{H}_0^ρ ; this is obtained by functional analytical techniques. The kernel of such restriction is obtained as a special case when the eigenvalue is zero. It is convenient to treat this subject using an L^2 -approach (this facilitates the computation of adjoint maps), and for this one must enter in the realm of unbounded operators. The following notation will be used:

$$\mathfrak{K} = L^2([0, 1], \mathbb{C}^n),$$

$$\mathfrak{K}_* = \left\{ V \in \mathfrak{K} : g(V, Y) - 2 \int_0^t g(V, Y') \, ds = 2t \int_0^1 g(V, Y') \, ds \text{ a.e. on } [0, 1] \right\},$$

and:

$$\mathfrak{K}_0 = \left\{ V \in \mathfrak{K} : g(V, Y) - 2 \int_0^t g(V, Y') \, ds = 0 \text{ a.e. on } [0, 1] \right\}.$$

We want to describe the orthogonal subspaces to \mathfrak{K}_* and \mathfrak{K}_0 in \mathfrak{K} relatively to the inner product

$$(3.1) \quad \langle V, W \rangle_{\mathfrak{K}} = \int_0^1 g_t^{(r)}(V, W) \, dt.$$

The subspaces \mathfrak{K}_* and \mathfrak{K}_0 are the kernels respectively of the bounded linear operators $T_* : \mathfrak{K} \rightarrow L^2([0, 1]; \mathbb{C})$ given by

$$(3.2) \quad T_*(V)(t) = g(V(t), Y(t)) - 2 \int_0^t g(V, Y') \, ds - 2t \int_0^1 g(V, Y') \, ds,$$

and $T_0 : \mathfrak{K} \rightarrow L^2([0, 1]; \mathbb{C})$ given by

$$(3.3) \quad T_0(V)(t) = g(V(t), Y(t)) - 2 \int_0^t g(V, Y') \, ds.$$

Lemma 3.1. *The operators T_* and T_0 have closed (and finite codimensional) image.*

Proof. Consider the operators $\tilde{T}_*, \tilde{T}_0 : L^2([0, 1], \mathbb{C}) \rightarrow L^2([0, 1], \mathbb{C})$ defined respectively by $\tilde{T}_*(\mu) = T_*(\mu \cdot Y)$ and $\tilde{T}_0(\mu) = T_0(\mu \cdot Y)$. Clearly, $\text{Im}(\tilde{T}^*) \subset \text{Im}(T^*)$ and $\text{Im}(\tilde{T}^0) \subset \text{Im}(T^0)$; it suffices to show that \tilde{T}_* and \tilde{T}_0 have finite codimensional closed

image.¹ Now, it is easy to see that both \widetilde{T}_* and \widetilde{T}_0 are Fredholm operators of index zero; namely, they are compact perturbations of the isomorphism $L^2([0, 1], \mathbb{C}) \ni \mu \mapsto \mu \cdot g(Y, Y) \in L^2([0, 1], \mathbb{C})$. \square

Keeping in mind (2.9), the adjoint operators $(T_*)^*$ and $(T_0)^*$ can be easily computed as:

$$(3.4) \quad (T_*)^*(\phi)(t) = \phi(t) \cdot (A(t, 1)Y(t)) - 2(A(t, 1)Y'(t)) \cdot \int_t^1 \phi(s) ds \\ + 2(A(t, 1)Y'(t)) \cdot \int_0^1 (1-s)\phi(s) ds$$

and

$$(3.5) \quad (T_0)^*(\phi)(t) = \phi(t) \cdot (A(t, 1)Y(t)) - 2(A(t, 1)Y'(t)) \cdot \int_t^1 \phi(s) ds.$$

Since T_* and T_0 have closed image, then also the adjoints $(T_*)^*$ and $(T_0)^*$ have closed image, and

$$\mathfrak{K}_*^\perp = \text{Ker}(T_*)^\perp = \text{Im}((T_*)^*), \quad \mathfrak{K}_0^\perp = \text{Ker}(T_0)^\perp = \text{Im}((T_0)^*).$$

The following corollary follows straightforward.²

Corollary 3.2. *The orthogonal space \mathfrak{K}_*^\perp in \mathfrak{K} is:*

$$(3.6) \quad \mathfrak{K}_*^\perp = \{h'' \cdot AY + 2h' \cdot AY' : h \in H^2([0, 1]; \mathbb{C}) \cap H_0^1([0, 1]; \mathbb{C})\}$$

and the orthogonal space \mathfrak{K}_0^\perp in \mathfrak{K} is

$$(3.7) \quad \mathfrak{K}_0^\perp = \{h' \cdot AY + 2h \cdot AY' : h \in H^1([0, 1]; \mathbb{C}) \text{ and } h(1) = 0\}.$$

Proof. It follows easily from the preceding observations, keeping in mind that the maps $H^2([0, 1]; \mathbb{C}) \cap H_0^1([0, 1]; \mathbb{C}) \ni h \mapsto h'' \in L^2([0, 1], \mathbb{C})$ and $\{h \in H^1([0, 1]; \mathbb{C}) : h(1) = 0\} \ni h \mapsto h' \in L^2([0, 1], \mathbb{C})$ are isomorphisms. \square

The bilinear form I_1 (defined in (2.7)) is represented in $L^2([0, 1]; \mathbb{C}^n)$ with respect to the inner product (3.1) by the unbounded self-adjoint operator:

$$(3.8) \quad \mathfrak{I}(V) = -AV'' + ARV$$

densely defined on the subspace $D = H^2([0, 1]; \mathbb{C}^n) \cap \mathcal{H}^\rho(1)$.

By an eigenvalue of the restriction of I_1 to $\mathfrak{K}_* \cap \mathcal{H}^\rho(1)$ we will mean a complex number λ_* such that there is a non-zero $V_* \in \mathfrak{K}_* \cap \mathcal{H}^\rho(1)$ satisfying

$$(3.9) \quad I_1(V_*, W) = \lambda_* \cdot \int_0^1 g_t^{(r)}(V_*, W) dt$$

for every $V, W \in \mathfrak{K}_*$. Equivalently, λ_* is an eigenvalue of the restriction of I_1 to $\mathfrak{K}_* \cap \mathcal{H}^\rho(1)$ if there exists $V_* \in \mathfrak{K}_* \cap H^2([0, 1]; \mathbb{C}^n) \cap \mathcal{H}^\rho(1)$ such that

$$(3.10) \quad \mathfrak{I}(V_*) - \lambda_* \cdot V_* \in \mathfrak{K}_*^\perp.$$

Proposition 3.3. *A vector $V_* \in \mathfrak{K}_* \cap \mathcal{H}^\rho(1)$ is an eigenvector for the restriction of I_1 to $\mathfrak{K}_* \cap \mathcal{H}^\rho(1)$ with eigenvalue $\lambda_* \in \mathbb{C}$ if and only if $V_* \in H^2([0, 1]; \mathbb{C}^n) \cap \mathcal{H}^\rho(1)$ and it satisfies*

$$(3.11) \quad -V_*'' + RV_* - \lambda_* \cdot A^{-1}V_* = h'' \cdot Y + 2h' \cdot Y',$$

where h is the unique map in $H^2([0, 1]; \mathbb{C}) \cap H_0^1([0, 1]; \mathbb{C})$ satisfying

$$(3.12) \quad \lambda_* \cdot g_t^{(r)}(V_*, Y) = [h' \cdot g(Y, Y)]'.$$

¹Recall that given a closed finite codimensional subspace X of a Hilbert space H , then any subspace $Y \subset H$ that contains X is closed (and finite codimensional).

²In the sequel, we will use the symbol A to mean $A(\cdot, 1)$.

Proof. By a boot-strap argument we see that if V_* is an eigenvector then it is differentiable. Using equations (3.8) and (3.10) we conclude easily that V_* satisfies (3.11) and (3.12) if and only if V_* is an eigenvector with λ_* as eigenvalue. Moreover, these equations imply that $g(V''_*, Y) = g(V_*, Y'')$, so that $V_* \in \mathfrak{R}_*$. \square

We obtain an analogous result for \mathfrak{R}_0 .

Proposition 3.4. *A vector $V_0 \in \mathfrak{R}_0 \cap \mathcal{H}^\rho(1)$ is an eigenvector for the restriction of I_1 to $\mathfrak{R}_0 \cap \mathcal{H}^\rho(1)$ with eigenvalue $\lambda_0 \in \mathbb{C}$ if and only if $V_0 \in H^2([0, 1]; \mathbb{C}^n) \cap \mathcal{H}^\rho(1)$, $g(V'_0(0), Y(0)) - g(V_0(0), Y'(0)) = 0$ and the following differential equation is satisfied*

$$(3.13) \quad -V''_0 + RV_0 - \lambda_0 \cdot A^{-1}V_0 = h' \cdot Y + 2h \cdot Y',$$

where

$$(3.14) \quad h = -\frac{\lambda_0}{g(Y, Y)} \int_t^1 g_s^{(r)}(V_0, Y) ds.$$

Proof. Similar to Proposition 3.3. \square

Setting $\lambda_0 = 0$ in Proposition 3.4, one obtains that the elements in the kernel of the restriction of I_1 to $\mathfrak{R}_0 \cap \mathcal{H}^\rho(1)$ are solutions of the Morse–Sturm system (2.1). This statement is made more precise in the following:

Corollary 3.5. $\text{Ker}[I_1|_{\mathcal{H}_0^1(1) \times \mathcal{H}_0^1(1)}] = \text{Ker}[I_1|_{\mathcal{H}_*^1(1) \times \mathcal{H}_*^1(1)}] \cap \mathcal{H}_0^1(1)$, while for $\rho \neq 1$, $\text{Ker}[I_1|_{\mathcal{H}_0^\rho(1) \times \mathcal{H}_0^\rho(1)}] = \text{Ker}[I_1|_{\mathcal{H}_*^\rho(1) \times \mathcal{H}_*^\rho(1)}]$. *In particular:*

$$(3.15) \quad \nu_0(1, 1) \leq \nu_*(1, 1) \leq \nu_0(1, 1) + 1, \quad \nu_0(\rho, 1) = \nu_*(\rho, 1) \text{ for } \rho \neq 1.$$

Proof. For $\lambda_0 = 0$, equation (3.13) is the Morse–Sturm system (2.1); the conclusion follows easily from Corollary 2.2. The second inequality in (3.15) follows from the fact that $\mathcal{H}_0^\rho(1)$ has codimension 1 in $\mathcal{H}_*^\rho(1)$. \square

Corollary 3.6. *If $\mathcal{A} \subset \mathbb{S}^1 \setminus \{1\}$ is a connected subset that does not contain elements in the spectrum of \mathfrak{P} , then the map $\lambda_0(\rho, 1)$ is constant on \mathcal{A} .*

Proof. The assumption is that $\nu_*(\rho, 1)$ vanishes on \mathcal{A} , which by Corollary 3.5 implies that also $\nu_0(\rho, 1)$ vanishes on \mathcal{A} . Corollary 2.11 concludes the thesis. \square

4. ON THE INDEX SEQUENCES

4.1. A Fourier theorem. In the following, given $\rho \in \mathbb{S}^1$ and $w \in \mathcal{H}_*^\rho(1)$, it will be useful to consider the (continuous) extension $w : \mathbb{R} \rightarrow \mathbb{C}^n$ of w defined by:

$$(4.1) \quad w(t + N) = \rho^N T^{-N} w(t), \quad \forall t \in [0, 1[, \forall N \in \mathbb{Z}.$$

Observe that such extension does not satisfy $g(w', Y) - g(w, Y') = \text{const.}$ in \mathbb{R} , unless $\rho = 1$ or $w \in \mathcal{H}_0^\rho(1)$.

Proposition 4.1. *For all $N \geq 1$, set $\omega = e^{2\pi i/N}$ and define the map*

$$\Psi_N : \mathcal{H}_*^1(N) \rightarrow \left[\bigoplus_{k=1}^{N-1} \mathcal{H}_0^{\omega^k}(1) \right] \oplus \mathcal{H}_*^1(1)$$

by:

$$V \mapsto (V_1, \dots, V_N)$$

where

$$(4.2) \quad V_k(t) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} T^j V \left(\frac{t+j}{N} \right),$$

for all $t \in [0, 1[$. Then, Ψ_N is a linear isomorphism, whose inverse

$$\Upsilon_N : \left[\bigoplus_{k=1}^{N-1} \mathcal{H}_0^{\omega^k}(1) \right] \oplus \mathcal{H}_*^1(1) \longrightarrow \mathcal{H}_*^1(N)$$

is given by $(V_1, \dots, V_N) \mapsto V$, where³

$$(4.3) \quad V(t) = \sum_{k=1}^N V_k(tN),$$

for all $t \in [0, 1]$. The isomorphism Ψ_N carries $\mathcal{H}_0^1(N)$ onto the direct sum $\bigoplus_{k=1}^N \mathcal{H}_0^{\omega^k}(1)$.

Proof. The proof is a matter of direct calculations, based on a repeated use of the identity:

$$\sum_{r=0}^{N-1} \omega^{sr} = \begin{cases} 0, & \text{if } s \not\equiv 0 \pmod{N}; \\ N, & \text{if } s \equiv 0 \pmod{N}. \end{cases}$$

The details of the computations are as follows. Clearly, Ψ_N is linear and bounded. Let us show that it is well defined, i.e., that for all $k = 1, \dots, N-1$, the map V_k is in $\mathcal{H}_0^{\omega^k}(1)$ and that $V_N \in \mathcal{H}_*^{\omega^k}(1)$. We compute:

$$\begin{aligned} \omega^{-k} T V_k(1) &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-k(j+1)} T^{j+1} V\left(\frac{j+1}{N}\right) = \frac{1}{N} \sum_{j=1}^N \omega^{-kj} T^j V(j/N) \\ &= \frac{1}{N} \left(\sum_{j=1}^{N-1} \omega^{-kj} T^j V(j/N) + T^N V(1) \right) = \frac{1}{N} \left(\sum_{j=1}^{N-1} \omega^{-kj} T^j V(j/N) + V(0) \right) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} T^j V(j/N) = V_k(0). \end{aligned}$$

Moreover, for $t \in [0, 1[$, setting $s_j = (t + j)/N$, $j = 0, \dots, N-1$ and recalling formula (2.5):

$$\begin{aligned} g(V'_k(t), Y(t)) - g(V_k(t), Y'(t)) &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} \left[\frac{1}{N} g(T^j V'(\frac{t+j}{N}), Y(t)) - g(T^j V(\frac{t+j}{N}), Y'(t)) \right] \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} \left[\frac{1}{N} g(T^j V'(s_j), Y_N(s_j - \frac{j}{N})) - \frac{1}{N} g(T^j V(s_j), Y'_N(s_j - \frac{j}{N})) \right] \\ &= \frac{1}{N^2} \sum_{j=0}^{N-1} \omega^{-kj} [g(T^j V'(s_j), T^j Y_N(s_j)) - g(T^j V(s_j), T^j Y'_N(s_j))] \\ &= \frac{1}{N^2} \sum_{j=0}^{N-1} \omega^{-kj} [g(V'(s_j), Y_N(s_j)) - g(V(s_j), Y'_N(s_j))] \\ &= \frac{C_V}{N^2} \sum_{j=0}^{N-1} \omega^{-kj} = \begin{cases} 0, & \text{if } k < N; \\ \frac{1}{N} C_V, & \text{if } k = N. \end{cases} \end{aligned}$$

This proves that $V_k \in \mathcal{H}_0^{\omega^k}(1)$ for $k < N$ and that $V_N \in \mathcal{H}_*^1(1)$. In order to conclude the proof it remains to verify that (4.3) defines an inverse for Ψ_N . This is also a straightforward calculation. Given $V \in \mathcal{H}_*^1(N)$, then:

$$\frac{1}{N} \sum_{k=1}^N \sum_{j=0}^{N-1} \omega^{-kj} T^j V\left(\frac{tN+j}{N}\right) = \frac{1}{N} \sum_{j=0}^{N-1} T^j V\left(\frac{tN+j}{N}\right) \left(\sum_{k=1}^N \omega^{-kj} \right) = V(t),$$

³In equation (4.3) we are assuming that all the V_k 's have been extended to \mathbb{R} as in (4.1). An immediate calculation shows that if V and the V_k 's are extended to \mathbb{R} according to (4.1), then equality (4.2) holds for all $t \in \mathbb{R}$.

which proves that $\Upsilon_N \circ \Psi_N$ is the identity.

Conversely, given $(V_1, \dots, V_N) \in \left[\bigoplus_{k=1}^{N-1} \mathcal{H}_0^{\omega^k}(1) \right] \oplus \mathcal{H}_*^1(1)$, $k \in \{1, \dots, N\}$ and $t \in [0, 1[$:

$$\begin{aligned} \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} T^j \sum_{l=1}^N V_l(t+j) &= \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-kj} \sum_{l=1}^N \omega^{lj} V_l(t) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{l=1}^N \omega^{j(l-k)} V_l(t) \\ &= \frac{1}{N} \sum_{l=1}^N V_l(t) \left[\sum_{j=0}^{N-1} \omega^{j(l-k)} \right] = V_k(t), \end{aligned}$$

which proves that $\Psi_N \circ \Upsilon_N$ is the identity. This concludes the proof. \square

Finally, the desired result on the index sequences:

Proposition 4.2 (Fourier theorem). *For all $N \geq 1$, the following identities hold:*

$$\begin{aligned} \lambda_*(1, N) &= \lambda_*(1, 1) + \sum_{k=1}^{N-1} \lambda_0(\omega^k, 1), \\ \lambda_0(1, N) &= \sum_{k=1}^N \lambda_0(\omega^k, 1), \end{aligned}$$

where $\omega = e^{2\pi i/N}$.

Proof. The result is obtained by showing that, given $V_k, W_k \in \mathcal{H}_0^{\omega^k}(1)$, $k = 1, \dots, N-1$, $V_N, W_N \in \mathcal{H}_*^1(1)$, and setting $V = \Psi_N^{-1}(V_1, \dots, V_N)$, $W = \Psi_N^{-1}(W_1, \dots, W_N)$, the following identity holds:

$$I_N(V, W) = \sum_{k=1}^N I_1(V_k, W_k).$$

This is obtained by a direct calculation, keeping in mind that:

- $V_k(s+l-1) = \omega^{k(l-1)} T^{1-l} V_k(s)$, $V'_k(s+l-1) = \omega^{k(l-1)} T^{1-l} V'_k(s)$;
- $W_k(s+l-1) = \omega^{k(l-1)} T^{1-l} W_k(s)$, $W'_k(s+l-1) = \omega^{k(l-1)} T^{1-l} W'_k(s)$;
- $R(s+l-1) = T^{1-l} R(s) T^{l-1}$,

for all $s \in [0, 1]$. Then:

$$\begin{aligned} I_N(V, W) &= N^2 \sum_{k,r=1}^N \int_0^1 \left[g(V'_k(tN), W'_r(tN)) + g(R(tN)V_k(tN), W_r(tN)) \right] dt \\ &= N^2 \sum_{k,r=1}^N \sum_{l=1}^N \int_{\frac{l-1}{N}}^{\frac{l}{N}} \left[g(V'_k(tN), W'_r(tN)) + g(R(tN)V_k(tN), W_r(tN)) \right] dt \\ &= N \sum_{k,r=1}^N \sum_{l=1}^N \int_{l-1}^l \left[g(V'_k(s), W'_r(s)) + g(R(s)V_k(s), W_r(s)) \right] ds \\ &= N \sum_{k,r,l=1}^N \int_0^1 \left[g(V'_k(s+l-1), W'_r(s+l-1)) + g(R(s+l-1)V_k(s+l-1), W_r(s+l-1)) \right] ds \\ &= N \sum_{k,r=1}^N \sum_{l=1}^N \omega^{(k-r)(l-1)} \int_0^1 \left[g(V'_k(s), W'_r(s)) + g(R(s)V_k(s), W_r(s)) \right] ds \\ &= \sum_{k=1}^N \int_0^1 \left[g(V'_k(s), W'_k(s)) + g(R(s)V_k(s), W_k(s)) \right] ds = \sum_{k=1}^N I_1(V_k, W_k), \end{aligned}$$

which concludes the proof. \square

4.2. On the jumps of the index function. The question of determining the value of the jumps of the index function $\rho \mapsto \lambda_0(\rho, 1)$ is rather involved, and it will be treated only marginally in this subsection. Let us start with a simple observation on the index of continuously varying essentially positive symmetric bilinear forms, whose proof is omitted:

Lemma 4.3. *Let $B : \mathfrak{H} \times \mathfrak{H} \rightarrow \mathbb{C}$ be a Fredholm Hermitian form on the complex Hilbert space \mathfrak{H} , and let $\{\mathcal{D}^t\}_{t \in I}$ be a continuous family of closed subspaces of \mathfrak{H} such that the restriction B_t of B to $\mathcal{D}^t \times \mathcal{D}^t$ is essentially positive for all $t \in I$. If t_0 is an isolated instant in the interior of I such that B_{t_0} is nondegenerate, then for $\varepsilon > 0$ small enough:*

$$|n_-(B_{t_0+\varepsilon}) - n_-(B_{t_0-\varepsilon})| \leq \dim(\text{Ker}(B_{t_0})). \quad \square$$

Corollary 4.4. *Let $e^{2\pi i \tilde{\theta}} \in \mathbb{S}^1 \setminus \{1\}$ be a discontinuity point for the map $\rho \mapsto \lambda_0(\rho, 1)$. Then:*

$$\left| \lim_{\theta \rightarrow 0^+} [\lambda_0(e^{2\pi i(\tilde{\theta}+\theta)}, 1) - \lambda_0(e^{2\pi i(\tilde{\theta}-\theta)}, 1)] \right| \leq \nu_0(e^{2\pi i \tilde{\theta}}, 1).$$

If Y is not a singular solution of (2.1), the same conclusion holds also for $\tilde{\theta} = 0$.

Proof. It follows immediately from Proposition 2.6 (or Corollary 2.9 in the singular case), Proposition 2.10 and Lemma 4.3. \square

Under a certain nondegeneracy assumption, the jump of the index function at a discontinuity point can be computed in terms of a finite dimensional reduction (compare with [8, Theorem IV, p. 180]). This finite dimensional reduction is rather technical, and we will only sketch its construction. Given $e^{2\pi i \tilde{\theta}} \in \mathfrak{s}(\mathfrak{P}_0) \cap \mathbb{S}^1$, let us define a Hermitian form $B_{\tilde{\theta}}$ on the finite dimensional vector space $N_{\tilde{\theta}} = \text{Ker}(\mathfrak{P}_0 - e^{2\pi i \tilde{\theta}})$ as follows. Identify vectors $v \in N_{\tilde{\theta}}$ with functions $V \in \mathcal{H}_0^{e^{2\pi i \tilde{\theta}}}(1)$ in the kernel of I_1 (use Proposition 2.12 and Corollary 3.5) and, given one such V , choose an arbitrary C^1 -map $\mathfrak{V} :]-\varepsilon, \varepsilon[\rightarrow \mathcal{H}$ with $\mathfrak{V}(s) \in \mathcal{H}_0^{e^{2\pi i(\tilde{\theta}+s)}}(1)$ for all s , and such that $\mathfrak{V}(0) = V$. Finally, denote by $P_{\tilde{\theta}} : \mathcal{H} \rightarrow \mathcal{H}_0^{e^{2\pi i \tilde{\theta}}}(1)$ the orthogonal projection, and set:

$$B_{\tilde{\theta}}(v, w) = I_1(P_{\tilde{\theta}}\mathfrak{V}'(0), \mathfrak{V}(0)) + I_1(\mathfrak{V}(0), P_{\tilde{\theta}}\mathfrak{W}'(0));$$

it is not hard to show that $B_{\tilde{\theta}}$ is well defined, i.e., that the right hand side in the above formula does not depend on the choice of the C^1 -maps \mathfrak{V} and \mathfrak{W} .

Proposition 4.5. *Let $e^{2\pi i \tilde{\theta}} \in \mathbb{S}^1$ be a discontinuity point for $\rho \mapsto \lambda_0(\rho, 1)$ (with $e^{2\pi i \tilde{\theta}} \neq 1$ if Y is a singular solution of (2.1)). Then, if $B_{\tilde{\theta}}$ is nondegenerate, for $\theta > 0$ small enough:*

$$\lambda_0(e^{2\pi i(\tilde{\theta}+\theta)}, 1) - \lambda_0(e^{2\pi i(\tilde{\theta}-\theta)}, 1) = -\text{signature}(B_{\tilde{\theta}}).$$

It is clear from our construction that the value of the jump of $\lambda_0(\rho, 1)$ at a discontinuity ρ_0 can be computed as an algebraic count of the eigenvalues through zero of the path $\rho \mapsto \mathcal{T}_\rho$ of self-adjoint Fredholm operators representing the index form I_1 in $\mathcal{H}_0^\rho(1)$ as ρ runs in the arc $\mathcal{A}_{\rho_0} = \{e^{2\pi i \theta} \rho_0, \theta \in [-\varepsilon, \varepsilon]\}$. Technically speaking, this is the so-called *spectral flow* of the path $\mathcal{A}_{\rho_0} \ni \rho \mapsto \mathcal{T}_\rho$, see for instance [23] for details on the spectral flow. By Proposition 2.6 (or Proposition 2.8 in the singular case), the path \mathcal{T}_ρ is analytic, in which case higher order methods are available in order to compute the value of the spectral flow at each degeneracy instant (see [11] for details). The result in Proposition 4.5 is a special case of this method.

5. CLOSED GEODESICS IN STATIONARY LORENTZIAN MANIFOLDS

5.1. Closed geodesics. Let us consider a Lorentzian manifold (M, \mathfrak{g}) , with $\dim(M) = n + 1$, endowed with a timelike Killing vector field K ; let ∇ be the Levi-Civita connection of \mathfrak{g} and $\mathcal{R}(X, Z) = [\nabla_X, \nabla_Z] - \nabla_{[X, Z]}$ its curvature tensor (see [6, 15, 22] for details). An auxiliary Riemannian metric \mathfrak{g}^R on M is obtained by taking $\mathfrak{g}^R(v, w) = \mathfrak{g}(v, w) -$

$2\mathfrak{g}(v, \mathcal{K})\mathfrak{g}(w, \mathcal{K})\mathfrak{g}(\mathcal{K}, \mathcal{K})^{-1}$. Let $\gamma : [0, 1] \rightarrow M$ be a non constant closed geodesic in (M, \mathfrak{g}) and denote by $\tilde{\gamma} : \mathbb{R} \rightarrow M$ its periodic extension to the real line; for $N \geq 1$, let $\gamma^N : [0, 1] \rightarrow M$ denote the N -th iterate of γ , defined by $\gamma^N(s) = \tilde{\gamma}(Ns)$, $s \in [0, 1]$. There are two constants associated to γ : $E_\gamma = \mathfrak{g}(\dot{\gamma}, \dot{\gamma})$ and $c_\gamma = \mathfrak{g}(\dot{\gamma}, \mathcal{V})$; observe that, by causality, γ is spacelike, and thus $E_\gamma > 0$. Define a smooth vector field \mathcal{V} along γ by setting $\mathcal{V}(s) = \mathcal{K}(\gamma(s)) - c_\gamma E_\gamma^{-1} \dot{\gamma}(s)$; this is a periodic timelike Jacobi field along γ which is everywhere orthogonal to $\dot{\gamma}$. The index form \mathcal{I}_γ of γ , which is the second variation of the geodesic action functional at γ , is given by:

$$\mathcal{I}_\gamma(\mathcal{V}, \mathcal{W}) = \int_0^1 \left[\mathfrak{g}\left(\frac{D}{ds}\mathcal{V}, \frac{D}{ds}\mathcal{W}\right) + \mathfrak{g}(\mathcal{R}(\dot{\gamma}, \mathcal{V})\dot{\gamma}, \mathcal{W}) \right] ds;$$

here $\frac{D}{ds}$ denotes covariant differentiation along γ . \mathcal{I}_γ is a bounded symmetric bilinear form defined on the real Hilbert space \mathcal{H}^γ of all periodic vector fields along γ of Sobolev class H^1 that are *everywhere orthogonal* to $\dot{\gamma}$. Consider the following closed subspaces of \mathcal{H}^γ :

$$\begin{aligned} \mathcal{H}_*^\gamma &= \left\{ \mathcal{V} \in \mathcal{H}^\gamma : \mathfrak{g}\left(\frac{D}{ds}\mathcal{V}, \mathcal{V}\right) - \mathfrak{g}(\mathcal{V}, \frac{D}{ds}\mathcal{V}) \text{ is constant on } [0, 1] \right\}, \\ \mathcal{H}_0^\gamma &= \left\{ \mathcal{V} \in \mathcal{H}^\gamma : \mathfrak{g}\left(\frac{D}{ds}\mathcal{V}, \mathcal{V}\right) - \mathfrak{g}(\mathcal{V}, \frac{D}{ds}\mathcal{V}) = 0 \text{ a.e. on } [0, 1] \right\}. \end{aligned}$$

Elements in \mathcal{H}_*^γ are variational vector fields along γ corresponding to variations of γ by curves μ for which the quantity $\mathfrak{g}(\dot{\mu}, \mathcal{V})$ is constant; similarly, elements of \mathcal{H}_0^γ correspond to variations of γ by curves μ for which $\mathfrak{g}(\dot{\mu}, \mathcal{V})$ is equal to the constant c_γ . The restrictions of \mathcal{I}_γ to \mathcal{H}_*^γ and to \mathcal{H}_0^γ have finite index (see [7] for details); they will be denoted respectively by $\mu(\gamma)$ and $\mu_0(\gamma)$, and called the *Morse index* and the *restricted Morse index* of γ . Since \mathcal{H}_0^γ has codimension 1 in \mathcal{H}_*^γ , then $\mu_0(\gamma) \leq \mu(\gamma) \leq \mu_0(\gamma) + 1$; we will denote by $\epsilon_\gamma \in \{0, 1\}$ the difference $\mu(\gamma) - \mu_0(\gamma)$. The *nullity* of γ , $n(\gamma)$ is defined as the dimension of the space of periodic Jacobi fields along γ that are everywhere orthogonal to $\dot{\gamma}$, or, equivalently, as the dimension of the kernel of \mathcal{I}_γ in \mathcal{H}^γ . Similarly, we will denote by $n_0(\gamma)$ the *restricted nullity* of γ , defined as the dimension of the kernel of the restriction of \mathcal{I}_γ to \mathcal{H}_0^γ . Let Σ be a hypersurface of M through $\gamma(0)$ which is orthogonal to $\dot{\gamma}(0)$; denote by $TM_{E_\gamma}|_\Sigma$ the restriction to Σ of the sphere bundle $\{v \in TM : g(v, v) = E_\gamma\}$. Let $\mathcal{P}_\Sigma : \mathcal{U}_\Sigma \rightarrow \mathcal{U}_\Sigma$ denote the Poincaré map of Σ , defined in a sufficiently small neighborhood \mathcal{U}_Σ of $\dot{\gamma}(0)$ in $TM_{E_\gamma}|_\Sigma$. Recall that \mathcal{P}_Σ preserves the symplectic structure inherited from TM (here one uses the metric \mathfrak{g} to induce a symplectic form from TM^* to TM), and that $\dot{\gamma}(0)$ is a fixed point of \mathcal{P}_Σ . The *linearized Poincaré map* of γ is the differential $\mathfrak{P}_\gamma = d\mathcal{P}_\Sigma(\dot{\gamma}(0)) : T_{\dot{\gamma}(0)}\mathcal{U}_\Sigma \rightarrow T_{\dot{\gamma}(0)}\mathcal{U}_\Sigma$. If one uses the horizontal distribution of the connection ∇ to identify $T_{\dot{\gamma}(0)}(TM)$ with the direct sum $T_{\dot{\gamma}(0)}M \oplus T_{\dot{\gamma}(0)}M$, then $T_{\dot{\gamma}(0)}\mathcal{U}_\Sigma$ is identified with $\dot{\gamma}(0)^\perp \oplus \dot{\gamma}(0)^\perp$, and $\mathfrak{P}_\gamma(v, w) = (J(1), \frac{D}{ds}J(1))$, where J is the unique Jacobi field along γ satisfying the initial conditions $J(0) = v$ and $\frac{D}{ds}J(0) = w$. The closed geodesic γ will be called *singular* if the covariant derivative $\frac{D}{ds}\mathcal{K}$ of the restriction of the Killing field \mathcal{K} along γ is pointwise multiple of the orthogonal projection of \mathcal{K} onto $\dot{\gamma}^\perp$. This condition is the same as assuming that the covariant derivative $\frac{D}{ds}\mathcal{V}$ of the Jacobi field \mathcal{V} is pointwise multiple of \mathcal{V} . Observe that, by Lemma 2.7, if γ is singular then $\epsilon_\gamma = 0$.

We will consider the complexification of the Hilbert spaces defined above, as well as the complexification of the linear maps and the sesquilinear extension of \mathcal{I}_γ ; these complexified objects will be denoted by the same symbols as their real counterparts.

5.2. Geodesics and Morse–Sturm systems. For $t \in [0, 1]$, denote by $\mathbf{P}_t : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M$ the parallel transport; observe that \mathbf{P}_t carries $\dot{\gamma}(0)^\perp$ isomorphically onto $\dot{\gamma}(t)^\perp$. We choose an isomorphism $\phi_0 : \mathbb{R}^n \rightarrow \dot{\gamma}(0)^\perp$, and we denote by $\phi_t : \mathbb{R}^n \rightarrow \dot{\gamma}(t)^\perp$ the isomorphism $\mathbf{P}_t \circ \phi_0$. Finally, set $T : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$, $T = \phi_0^{-1} \circ \mathbf{P}_1 \circ \phi_0 = \phi_0^{-1} \phi_1$. Consider the following data to build up a Morse–Sturm system as described in Section 2. Let g be the nondegenerate symmetric bilinear form on \mathbb{R}^n given by the pull-back $\phi_0^*\mathfrak{g}$; since the

parallel transport is an isometry, then $g = \phi_t^* g$ for all $t \in [0, 1]$, and T is g -preserving.⁴ Define $R(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $R(t) = \phi_t^{-1} \circ [\mathcal{R}(\dot{\gamma}(t), \cdot) \gamma(t)] \circ \phi_t$; since \mathcal{R} is g -symmetric, then R is g -symmetric. Since γ is periodic, $\mathcal{R}(\dot{\gamma}(0), \cdot) \dot{\gamma}(0) = \mathcal{R}(\dot{\gamma}(1), \cdot) \dot{\gamma}(1)$, and thus $T^{-1}R(0)T = R(1)$. Again, we will consider complexifications of these objects, that will be denoted by the same symbols as their real counterparts.

Using the isomorphisms $(\phi_s)_{s \in [0, 1]}$, from a map $V : [0, 1] \rightarrow \mathbb{C}^n$ one obtains a vector field \mathcal{V} along γ which is orthogonal to $\dot{\gamma}$, defined by $\mathcal{V}(s) = \phi_s(V(s))$; the periodicity condition $\mathcal{V}(0) = \mathcal{V}(1)$ corresponds to the condition $TV(1) = V(0)$. An immediate computation shows that the map $V \mapsto \mathcal{V}$, denoted by Ψ , carries the space of solutions of the Morse–Sturm system (2.1) to the space of Jacobi fields along γ that are everywhere orthogonal to $\dot{\gamma}$; define Y as $\Psi^{-1}(\mathcal{V})$, where \mathcal{V} is the orthogonal timelike Jacobi field along γ defined above, so that Y satisfies (e) in Subsection 2.1. It is also immediate to see that γ is a singular closed geodesic as defined in Subsection 5.1 exactly when Y is a singular solution of the Morse–Sturm system (2.1).

5.1. Example. An important class of examples of singular closed geodesics can be obtained by considering *static* Lorentzian manifolds (M, g) , i.e., Lorentzian manifolds admitting a timelike Killing vector field \mathcal{K} whose orthogonal distribution \mathcal{K}^\perp is integrable. Every integral submanifold of \mathcal{K}^\perp is a totally geodesic submanifold of M ; every closed geodesic in M which is orthogonal to \mathcal{K} at some point is contained in one such integral submanifold. Moreover, if (M, g) is globally hyperbolic or if M is simply connected, then every closed geodesic in (M, g) is orthogonal to \mathcal{K} and therefore contained in an integral submanifold Σ of \mathcal{K}^\perp . Every such geodesic is singular. Namely, let $\{E_i(t)\}_i$ be a parallel frame of $T\Sigma$ along γ relatively to the Riemannian metric on Σ obtained by restriction of g . Since Σ is totally geodesic, then E_i is a parallel also in (M, g) ; and since $g(\mathcal{K}, E_i) = 0$, by differentiating we obtain $g(\frac{D}{ds}\mathcal{K}, E_i) = 0$ for all i . This shows that $\frac{D}{ds}\mathcal{K}$ is pointwise multiple of \mathcal{K} , i.e., γ is singular. It follows in particular that $\epsilon_\gamma = 0$ for all closed geodesic γ , more generally, the same conclusion holds when γ is contained in a totally geodesic hypersurface of M which is everywhere orthogonal to \mathcal{K} . Closed geodesics in compact static Lorentzian manifolds are studied in [24].

Using the isomorphism $\phi_0 \oplus \phi_0$, the restriction of the linearized Poincaré map \mathfrak{P}_γ to $\dot{\gamma}(0)^\perp \oplus \dot{\gamma}(0)^\perp$ is identified with the linear Poincaré map \mathfrak{P} of the Morse–Sturm system defined in Subsection 2.6. The restricted Poincaré map \mathfrak{P}_0 of the Morse–Sturm system correspond to the restriction of \mathfrak{P}_γ to the invariant subspace $\mathcal{E}_0 \subset \dot{\gamma}(0)^\perp \oplus \dot{\gamma}(0)^\perp$ consisting of pairs (v, w) such that $g(w, \mathcal{V}(\gamma(0))) - g(v, \frac{D}{ds}\mathcal{V}(\gamma(0))) = 0$.

Recalling the notations in Subsection 2.3, we have an isomorphism $\Psi : \mathcal{H}^1(1) \rightarrow \mathcal{H}^\gamma$ that carries the spaces $\mathcal{H}_*^1(1)$ and $\mathcal{H}_0^1(1)$ respectively onto \mathcal{H}_*^γ and \mathcal{H}_0^γ . Moreover, the pull-back by Ψ of the index form I_γ is the bilinear form I_1 defined in (2.7). In total analogy, the index form I_{γ^N} of the N -th iterate γ^N of γ corresponds to the index form I_N in (2.7). The indexes and the nullities of the geodesic γ are therefore related to the indexes and nullities of the Morse–Sturm system (Subsection 2.7) by:

$$\mu(\gamma^N) = \lambda_*(1, N), \quad \mu_0(\gamma^N) = \lambda_0(1, N), \quad n(\gamma^N) = \nu_*(1, N), \quad n_0(\gamma^N) = \nu_0(1, N).$$

If we define $\Lambda_\gamma, N_\gamma : \mathbb{S}^1 \rightarrow \mathbb{N}$ by setting:

$$\Lambda_\gamma(\rho) = \lambda_0(\rho, 1), \quad N_\gamma(\rho) = \nu_0(\rho, 1), \quad \rho \in \mathbb{S}^1,$$

we can state the central result of the paper:

Theorem 5.1. *For all $N \geq 1$, the following statements hold:*

$$(1) \quad \mu(\gamma^N) = \mu(\gamma) + \sum_{k=1}^{N-1} \Lambda_\gamma(e^{2\pi i k/N}) = \epsilon_\gamma + \sum_{k=1}^N \Lambda_\gamma(e^{2\pi i k/N}).$$

⁴It is interesting to observe here that T belongs to the connected component of the identity of $O(\mathbb{R}^n, g)$ precisely when the geodesic γ is orientation preserving, i.e., when the parallel transport \mathbf{P}_1 is orientation preserving.

- (2) $\mu_0(\gamma^N) = \sum_{k=1}^N \Lambda_\gamma(e^{2\pi i k/N})$.
- (3) If γ is non singular, the jumps of Λ_γ can only occur at points of the spectrum of \mathfrak{P}_γ that lie on the unit circle; if γ is singular a jump of Λ_γ can occur also at 1.
- (4) If γ is non singular and $e^{2\pi i \tilde{\theta}}$ is a discontinuity point of Λ_γ , then $\Lambda_\gamma(e^{2\pi i \tilde{\theta}}) \leq \lim_{\theta \rightarrow 0^\pm} \Lambda_\gamma(e^{2\pi i \tilde{\theta} + \theta})$. Moreover, the following estimate on the jump of Λ_γ holds:

$$\left| \lim_{\theta \rightarrow 0^+} \Lambda_\gamma(e^{2\pi i \tilde{\theta} + \theta}) - \lim_{\theta \rightarrow 0^-} \Lambda_\gamma(e^{2\pi i \tilde{\theta} + \theta}) \right| \leq N_\gamma(e^{2\pi i \tilde{\theta}}).$$

The same conclusion holds when γ is singular, under the additional assumption that $e^{2\pi i \tilde{\theta}} \neq 1$.

- (5) For all $N \geq 1$, $\epsilon_\gamma = \epsilon_{\gamma^N}$.

Proof. Parts (1) and (2) follow from Proposition 4.2. Part (3) follows from Corollary 2.11 and Proposition 2.12. The first statement in (4) is the lower semi-continuity property of the function $\rho \mapsto \lambda_0(\rho, 1)$ proved in (a) of Proposition 2.15 and the second one follows from Corollary 4.4; part (5) follows from (1) and (2). \square

5.3. Iteration formulas for the Morse index. Let us show that the sequence $\mu(\gamma^N)$ has linear growth in N :

Proposition 5.2. *Either $\mu(\gamma^N)$ is a constant sequence (equal to ϵ_γ), or the limit:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mu(\gamma^N) = \lim_{N \rightarrow \infty} \frac{1}{N} \mu_0(\gamma^N)$$

exists, it is finite and positive. In this case, its value is given by a sum of the type:

$$(5.1) \quad a_0 + \sum_{j=1}^K a_j \theta_j,$$

where a_i are integers, $a_0 > 0$, and $0 \leq \theta_1 < \theta_2 < \dots < \theta_K < 1$ are real numbers such that $e^{2\pi i \theta_j}$ belong to the spectrum of \mathfrak{P}_γ .

Proof. By part (3) of Theorem 5.1, Λ_γ has a finite number of discontinuities, thus it is Riemann integrable and the limit:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mu(\gamma^N) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \Lambda_\gamma(e^{2\pi i k/N})$$

equals the integral $\int_{\mathbb{S}^1} \Lambda_\gamma d\rho \geq 0$. Using (1) in Theorem 5.1, $\mu(\gamma^N)$ is constant if and only if Λ_γ vanishes identically (recall that Λ_γ is piecewise constant), hence if $\mu(\gamma^N)$ is not constant, $\int_{\mathbb{S}^1} \Lambda_\gamma d\rho > 0$. If $e^{2\pi i \theta_j}$ are the points of discontinuity of Λ_γ (which are necessarily points in the spectrum of \mathfrak{P}_γ by part (3) of Theorem 5.1), $j = 1, \dots, K$, setting:

$$(5.2) \quad \beta_j = \lim_{\theta \rightarrow 0^+} \Lambda_\gamma(e^{2\pi i(\theta_j + \theta)}) \in \mathbb{N} \setminus \{0\},$$

then the integral $\int_{\mathbb{S}^1} \Lambda_\gamma d\rho$ is given by the sum (5.1), where:

$$a_0 = \beta_K, \quad a_1 = \beta_K - \beta_1, \quad a_j = \beta_{j-1} - \beta_j, \quad j = 2, \dots, K-1. \quad \square$$

In order to apply equivariant Morse theory to the closed geodesic variational problem, one needs a somewhat finer estimate on the growth of the index by iteration. More precisely, it is needed a sort of *uniform superlinear growth* for the sequence $\mu(\gamma^N)$ (see [14, § 1]). The result of Proposition 5.2 can be improved as follows.

Proposition 5.3. *Either $\mu(\gamma^N)$ is a constant sequence, or there exist constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that:*

$$\mu(\gamma^{N+s}) - \mu(\gamma^N) \geq \alpha \cdot s + \beta$$

for all $N, s \in \mathbb{N}$.

Proof. As above, let $e^{2\pi i \theta_j}$ be the discontinuity points of the map Λ_γ , where $0 \leq \theta_1 < \theta_2 < \dots < \theta_K < 1$, set $\theta_{K+1} = \theta_1 + 1$, and define β_j as in (5.2). Note that, by (4) in Theorem 5.1, $\Lambda_\gamma(e^{2\pi i \theta_j}) \leq \beta_j$ for all j . If we denote by $\lfloor \cdot \rfloor$ the integer part function, the number of $(N+s)$ -th roots of unity that lie in the open arc $\{e^{2\pi i \theta} : \theta_j < \theta < \theta_{j+1}\}$ is at least $\lfloor (N+s)(\theta_{j+1} - \theta_j) \rfloor - 1$. Similarly, the number of N -th roots of unity that lie in the arc $\{e^{2\pi i \theta} : \theta_j \leq \theta < \theta_{j+1}\}$ is at most $\lfloor N(\theta_{j+1} - \theta_j) \rfloor + 1$. Thus, the following inequality holds:

$$\begin{aligned}
 (5.3) \quad \mu(\gamma^{N+s}) - \mu(\gamma^N) &= \sum_{k=1}^{N+s} \Lambda_\gamma(e^{2\pi i k/(N+s)}) - \sum_{k=1}^N \Lambda_\gamma(e^{2\pi i k/N}) \\
 &\geq \sum_{j=1}^{K+1} (\lfloor (N+s)(\theta_{j+1} - \theta_j) \rfloor - 1) \beta_j - \sum_{j=1}^{K+1} (\lfloor N(\theta_{j+1} - \theta_j) \rfloor + 1) \beta_j \\
 &\geq \sum_{j=1}^{K+1} (\lfloor (N+s)(\theta_{j+1} - \theta_j) \rfloor - \lfloor N(\theta_{j+1} - \theta_j) \rfloor) \beta_j - 2(K+1) \max \Lambda_\gamma \\
 &\geq \sum_{j=1}^{K+1} (\lfloor s(\theta_{j+1} - \theta_j) \rfloor - 1) \beta_j - 2(K+1) \max \Lambda_\gamma \\
 &\geq \sum_{j=1}^{K+1} (\lfloor s(\theta_{j+1} - \theta_j) \rfloor) \beta_j - 3(K+1) \max \Lambda_\gamma.
 \end{aligned}$$

The assumption that $\mu(\gamma^N)$ is not a constant sequence implies the existence of at least one $\theta_0 \in [0, 1[$ such that $\Lambda_\gamma(e^{2\pi i \theta_0}) > 0$; then $\theta_0 \in [\theta_{j_0}, \theta_{j_0+1}[$ for some j_0 , which implies $\beta_{j_0+1} - \beta_{j_0} > 0$ and $\beta_{j_0} > 0$. From (5.3) we therefore obtain:

$$\begin{aligned}
 \mu(\gamma^{N+s}) - \mu(\gamma^N) &\geq \lfloor s(\theta_{j_0+1} - \theta_{j_0}) \rfloor \cdot \beta_{j_0} - 3(K+1) \max \Lambda_\gamma \\
 &\geq s(\theta_{j_0+1} - \theta_{j_0}) \beta_{j_0} - \beta_{j_0} - 3(K+1) \max \Lambda_\gamma.
 \end{aligned}$$

This concludes the proof. \square

5.4. Hyperbolic geodesics. A closed geodesic $\gamma : [0, 1] \rightarrow M$ is said to be *hyperbolic* if the linearized Poincaré map \mathfrak{P}_γ has no eigenvalues on the unit circle. We say that a closed geodesic γ is *strongly hyperbolic* if, in addition, $\epsilon_\gamma = 0$. Observe that if γ is (strongly) hyperbolic, then γ^N is also (strongly) hyperbolic for all $N \geq 1$.

Lemma 5.4. *If γ is hyperbolic, then $\mu(\gamma^N) = \epsilon_\gamma + N \cdot \mu_0(\gamma)$ for all $N \geq 1$. If γ is strongly hyperbolic, then $\mu(\gamma^N) = \mu_0(\gamma^N) = N \cdot \mu_0(\gamma)$.*

Proof. Immediate using Theorem 5.1; here $\mu_0(\gamma)$ is the constant value of the function Λ_γ on the unit circle. \square

Let us assume that the Killing vector field \mathcal{K} is complete, and let us denote by $\psi_t : M \rightarrow M$ its flow, $t \in \mathbb{R}$, which consists of global isometries of (M, g) . We recall that two closed geodesics $\gamma_i : [0, 1] \rightarrow M$, $i = 1, 2$, are said to be *geometrically distinct* if there exists no $t \in \mathbb{R}$ such that $\psi_t \circ \gamma_1([0, 1]) = \gamma_2([0, 1])$. Let us denote by ΛM the free loop space of M , which consists of all closed curves $c : [0, 1] \rightarrow M$ of Sobolev class H^1 , endowed with the H^1 -topology. Moreover, given a spacelike hypersurface $S \subset M$, let \mathcal{N}_S be the subset of ΛM consisting of those curves c such that the quantity $g(\dot{c}, \mathcal{K})$ is constant

on $[0, 1]$, and with $c(0) \in S$. \mathcal{N}_S is a smooth, closed, embedded submanifold of ΛM ; let us recall from [7] (see also [9]) the following result:

Proposition 5.5. *Let (M, g) be a Lorentzian manifold endowed with a complete timelike Killing vector field and a compact Cauchy surface S . Then, there exists a closed geodesic in every connected component of ΛM ; more precisely, the inclusion $\mathcal{N}_S \hookrightarrow \Lambda M$ is a homotopy equivalence, and the geodesic functional $f(c) = \frac{1}{2} \int_0^1 g(\dot{c}, \dot{c}) ds$ is bounded from below and has a minimum point in every connected component of \mathcal{N}_S , which is a geodesic in (M, g) . The Morse index of a critical point γ of f in \mathcal{N}_S equals $\mu(\gamma)$. \square*

Recall that arc-connected components of ΛM correspond to conjugacy classes of the fundamental group $\pi_1(M)$; given one such component Λ_* , we will call *minimal* a closed geodesic γ in Λ_* , with $\gamma(0) \in S$, which is a minimum point for the restriction of f to the arc-connected component of \mathcal{N}_S containing γ . If M is not simply connected, Proposition 5.5 gives a multiplicity of minimal closed geodesics, however, there is no way of telling whether these geodesics are geometrically distinct. Let us recall that there is a continuous action of the orthogonal group $O(2)$ on the free loop space ΛM , obtained from the action of $O(2)$ on the parameter space S^1 . The geodesic functional f is invariant by this action. Let us also recall that the stabilizer of each point in ΛM is a finite cyclic subgroup of $SO(2)$, and thus the critical $O(2)$ -orbits of f are smooth submanifolds of ΛM that are diffeomorphic to two copies of the circle S^1 . Using the flow of the Killing field \mathcal{K} , one has also a free \mathbb{R} -action on ΛM given by $\mathbb{R} \times \Lambda M \ni (t, \gamma) \mapsto \psi_t \circ \gamma \in \Lambda M$, and f is invariant by this action. The quotient $\Lambda M / \mathbb{R}$ can be identified in an obvious way with the submanifold \mathcal{N}_S ; since the actions of \mathbb{R} and of $O(2)$ commute, one can define a continuous $O(2)$ -action on \mathcal{N}_S .

Inspired by a classical Riemannian result proved in [3], we give the following:

Proposition 5.6. *Under the hypotheses of Proposition 5.5, assume that there exists a non trivial element \mathfrak{a} in $\pi_1(M)$ satisfying the following:*

- *there exist integers $n \neq m$ such that the free homotopy classes generated by the conjugacy classes of \mathfrak{a}^n and \mathfrak{a}^m coincide;*
- *for all $N \geq 1$, every geodesic in the free homotopy class of \mathfrak{a}^N is strongly hyperbolic.*

Then, there are infinitely many geometrically distinct closed geodesics in (M, g) .

Proof. Let γ be a minimal hyperbolic geodesic in the connected component of ΛM determined by the free homotopy class of \mathfrak{a} ; then, γ^n and γ^m are freely homotopic, and so are γ^{nl} and γ^{ml} for all $l \geq 1$. Since γ is minimal, then $\mu(\gamma) = 0$, and since γ is hyperbolic, $\mu(\gamma^{nl}) = \mu(\gamma^{ml}) = 0$ for all l . As proved in [7], the geodesic action functional is bounded from below and it satisfies the Palais–Smale condition on \mathcal{N}_S ; the (strong) hyperbolicity assumption implies that f is an $O(2)$ -invariant Morse function on each arc-connected component of \mathcal{N}_S determined by the free homotopy class of some iterate of \mathfrak{a} . Since γ^{nl} and γ^{ml} are in the same arc-connected component of \mathcal{N}_S and they have index 0, a classical result of equivariant Morse theory (strong Morse relations, see [10, 21]) implies the existence of another critical orbit $O(2)c_l$, where $c_l \in \mathcal{N}_S$ is freely homotopic to γ^{nl} and γ^{ml} , and whose Morse index is equal to 1. Observe that distinct critical $O(2)$ -orbits $O(2)c_a$ and $O(2)c_b$ of f in \mathcal{N}_S determine geometrically distinct closed geodesics if and only if a and b are not iterate one of the other. By the strong hyperbolicity assumption, the iterate c_l^N has index equal to N for all $N \geq 1$; in particular, the c_l 's are pairwise geometrically distinct, which concludes the proof. \square

The question of existence and multiplicity of closed geodesics in stationary spacetimes seems more involved than in the Riemannian case, and there are few precedent results. We can cite for example [19] for the existence of a closed geodesic and the recent paper [7]

for a generalization of a Gromoll-Meyer type result. For the Riemannian case the main reference is [17].

6. FINAL REMARKS, CONJECTURES AND OPEN QUESTIONS

Hyperbolic closed geodesics that are singular are obviously strongly hyperbolic. In view of Example 5.1, in the case of static Lorentzian manifolds Proposition 5.6 reproduces the central result in [3]. On the other hand, the topological conditions on the fundamental group of the manifold allows the authors of [3] to establish their infinitude results for a *generic* collection of Riemannian metrics on the given manifold. This is based on a result due to Klingenberg and Takens [18] that, given a compact differential manifold M , the assumptions of the Birkhoff–Lewis symplectic fixed point theorem holds for the Poincaré map of every non hyperbolic closed geodesic for a C^4 -generic set of Riemannian metrics g on M . No such result is known in Lorentzian geometry; even more, it is not even known whether Lorentzian *bumpy metrics* are generic. Recall that a metric is bumpy if all its closed geodesics are nondegenerate; in the case of stationary Lorentzian metrics, such definition clearly needs to be adapted.

The genericity of Riemannian bumpy metrics on a compact manifold was proven by Abraham in [2]; a more recent elegant proof is given in [25]. The central point in [25] is that the Jacobi operator is strongly elliptic; a similar property is satisfied by the differential operator obtained from the second variation of the constrained variational problem in Proposition 5.5. This suggests the conjecture that, also in the case of stationary Lorentzian manifolds with a compact Cauchy surface, bumpy metrics are generic.

The recently developed theory of stationary Lorentzian closed geodesics and their iteration (see also [7]) suggests that a number of classical Riemannian results can be generalized to this context. For instance, one cannot avoid mentioning a possible extension to the stationary Lorentzian case of a result due to Bangert and Hingston [4]. The authors’ beautiful argument, based on Lusternik–Schnirelman theory, gives the existence of infinitely many closed geodesics in compact Riemannian manifolds whose fundamental group is infinite and abelian. We conjecture that the same result holds in the case of globally hyperbolic stationary Lorentzian manifolds. More results on the infinitude of closed Riemannian based on the study of the homology generated by a tower of iterates can be found in [5]. Research in this direction for stationary Lorentzian manifolds is being carried out, and it will be discussed in forthcoming papers.

Finally, we observe that a quite challenging task in the development of Morse theory for closed Lorentzian geodesics would be removing the stationarity assumption. In this case, one should deal with a truly strongly indefinite functional. Relations between its critical points and the homological properties of the free loop space must then be obtained by a more involved Morse theory based on a doubly infinite chain complex determined by the dynamics of the gradient flow (see [1] for the case of geodesics between fixed endpoints). In this case, the notion of Morse index has to be replaced by that of *spectral flow* for a path of Fredholm bilinear forms. We believe that the iteration results proven in this paper generalize to spectral flows.

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